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Invitation to Hilbert C^*-modules and Morita–Rieffel Equivalence

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1. Introduction

Hilbert C^*-modules play a fundamental role in modern theory of operator algebras and related fields. From the present perspective, one could distinguish the following main areas of application, which were initiated respectively by Rieffel (1973), Kasparov (1981), Woronowicz (1991) and Pimsner (1997): (I) Induced representations and Morita equivalence; (II) KK-theory; (III) C^*-algebraic quantum groups; and (IV) Universal C^*-algebras.

Topics (I)–(III) are well established and thoroughly discussed in monographs: see [5] for (I), [2] for (II) and [3] for (III). The present notes form an extended abstract from a series of lectures, whose main aim was to introduce elements of the theory of Hilbert C^*-modules in a form suitable for further studies on modern approach to noncommutative dynamics and universal C^*-algebras (IV).

2. Hilbert C^*-modules and adjointable maps

Hilbert C^*-modules over commutative C^*-algebras appeared first in the work of Kaplansky (1953). The rudiments of the theory for general C^*-algebras were elaborated in the PhD thesis of Paschke (1972). The idea behind the notion is simple: "generalize Hilbert spaces by replacing complex numbers with a general C^*-algebra".

Namely, let A be a C^*-algebra. A (right) *pre-Hilbert A-module* is a (right) A-module X equipped with a map $\langle \cdot, \cdot \rangle_A : X \times X \to A$ such that:

(1) $\langle x, ya + zb \rangle_A = \langle x, y \rangle_A \, a + \langle x, z \rangle_A \, b$ for any $x, y, z \in X$ and $a, b \in A$;
(2) $\langle x, y \rangle_A^* = \langle y, x \rangle_A$ for any $x, y \in X$;
(3) $\langle x, x \rangle_A \geq 0$ for any $x \in X$ (positivity in A);
(4) $\langle x, x \rangle_A = 0$ implies $x = 0$ for any $x \in X$.

The map $\langle x,y\rangle_A$ is called an *A-valued inner-product*. Generalizing standard arguments one can show that defining

$$\|x\| := \sqrt{\|\langle x,x\rangle_A\|}, \qquad x\in X,$$

the function $d(x,y)=\|x-y\|$ is a metric on X. We say that X is a (right) *Hilbert A-module* if it is complete with respect to d.[1] Then using an approximate unit $\{\mu_\lambda\}$ in A one can show that the formula $\lambda x := \lim_\lambda x(\lambda\mu_\lambda)$, $\lambda\in\mathbb{C}$, $x\in X$, defines scalar multiplication on X. In this way X becomes a *complex Banach space* and $\langle\cdot,\cdot\rangle_A : X\times X\to A$ a *sesqui-linear form.*

Example (Hilbert spaces). Hilbert $\mathbb{C}$-modules are Hilbert spaces.

Example (C^*-algebras). A C^*-algebra A may be treated as a Hilbert A-module equipped with the following natural right multiplication and A-valued inner product:

$$x\cdot a := xa, \qquad \langle x,y\rangle_A := x^*y, \qquad \text{for } x,y,a\in A.$$

Hilbert A-submodules of A correspond to closed right ideals in A.

Example (Concrete Hilbert A-modules). Let H be a Hilbert space. Let $A\subseteq B(H)$ be a C^*-subalgebra and $X\subseteq B(H)$ a closed subspace such that $XA\subseteq X$ and $X^*X\subseteq A$. Then X with operations inherited from $B(H)$ is a Hilbert A-module. Every Hilbert A-module can be represented in this form.

Example (Hilbert $C(M)$-modules = Vector bundles). Let $H=(\{H_t\}_{t\in M},\Gamma(H))$ be a continuous field of Hilbert spaces over a compact Hausdorff space M (i.e., $\{H_t\}_{t\in M}$ is a family of Hilbert spaces, $\Gamma(H)$ is a linear subspace of sections $M\ni t\mapsto x(t)\in H_t$ such that $M\ni t\mapsto\|x(t)\|$ is continuous, elements of $\Gamma(H)$ exhaust each space H_t, and $\Gamma(H)$ is maximal with these properties). Then $\Gamma(H)$ is a (right) Hilbert $C(M)$-module with the module action and a $C(M)$-valued sesqui-linear form given by:

$$(xa)(t) := a(t)x(t), \qquad \langle x,y\rangle_{C(M)}(t) := \langle x(t),y(t)\rangle,$$

$x\in\Gamma(H),\ a\in C(M),\ t\in M$. Every Hilbert $C(M)$-module is of the form described above.

Let X and Y be Hilbert A-modules. We say that a map $T:X\to Y$ is an *adjointable operator* if there exists a map $T^*:Y\to X$ such that

$$\langle Tx,y\rangle_A = \langle x,T^*y\rangle_A, \quad \text{for all } x\in X,\ y\in Y.$$

It follows then that both T and T^* are bounded $\mathbb{C}$-linear and A-linear operators. Moreover, T determines uniquely T^* and vice versa. In general, not every bounded (or even isometric) A-linear map is adjointable even when A is commutative. The set $\mathcal{L}(X,Y)$ of all adjointable operators from X to Y is a Banach space with respect to the operator norm. The space $\mathcal{L}(X):=\mathcal{L}(X,X)$ is a unital C^*-algebra

[1] Analogously one defines left Hilbert modules, as a left A-module equipped with an A-valued inner product which is A-linear with respect to the first variable.

with involution given by adjoint of an adjointable operator. For each $x \in X$, $y \in Y$, the map $\Theta_{x,y} : Y \to X$ defined by

$$\Theta_{x,y}(z) = x\langle y, z\rangle_A$$

is an adjointable operator with $\Theta^*_{x,y} = \Theta_{y,x}$. The elements of $\mathcal{K}(Y,X) := \overline{\text{span}}\{\Theta_{x,y} : x \in X, Y \in Y\} \subseteq \mathcal{L}(Y,X)$ are called (generalized) *compact operators* from Y to X. The set $\mathcal{K}(Y,X)$ is a Banach space and $\mathcal{K}(X) := \mathcal{K}(X,X)$ is an ideal in $\mathcal{L}(X)$.

Example (Hilbert spaces). If $A = \mathbb{C}$, then X and Y are Hilbert spaces and $\mathcal{L}(X,Y) = B(X,Y)$ are bounded operators and $\mathcal{K}(X,Y) = K(X,Y)$ are usual compact operators.

Example (C^*-algebras). If we treat a C^*-algebra A as a Hilbert A-module, then $\mathcal{K}(A) \cong A$ where $\Theta_{x,y} \mapsto xy^*$, $x, y \in A$. In particular, if $A = B(H)$ then $\mathcal{K}(A) \cong B(H)$. This shows that, in general, compact operators in the sense of Hilbert modules are not compact as operators between Banach spaces.

Example (Multiplier C^*-algebras). The multiplier algebra $M(A)$ of a C^*-algebra A is as a maximal essential unitization of A. For any Hilbert C^*-module X we have $M(\mathcal{K}(X)) \cong \mathcal{L}(X)$. In particular, $\mathcal{L}(A) \cong M(A)$.

3. C^*-correspondences

Let A, B be C^*-algebras. A *C^*-correspondence from A to B* is a (right) Hilbert B-module X equipped with a homomorphism $\phi_X : A \to \mathcal{L}(X)$ – left action of A on X. We write $b \cdot x := \phi_X(b)x$. We will treat C^*-correspondences as "generalized morphisms" between C^*-algebras. In particular, to denote that X is a C^*-correspondence from A to B we write $A \xrightarrow{X} B$. We also say that X is *non-degenerate* if $\phi_X(A)X = X$.

Example (Representations). Representations $\pi : A \to B(H)$ of a C^*-algebra A may be identified with C^*-correspondences $A \xrightarrow{H_\pi} \mathbb{C}$ from A to $\mathbb{C}$.

Example (Homomorphisms). If $\alpha : A \to B$ is a $*$-homomorphism we may treat it as a non-degenerate C^*-correspondence $A \xrightarrow{X_\alpha} B$ where $X_\alpha := \alpha(A)B$ is equipped with operations $a \cdot x := \alpha(a)x$, $x \cdot b := xb$, $\langle x, y\rangle_B := x^*y$ for all $x, y \in X_\alpha$, $a \in A$, $b \in B$.

Example (Concrete C^*-correspondences). Let $X \subseteq B(H)$ be a closed linear space and $A, B \subseteq B(H)$ be C^*-subalgebras such that $XB \subseteq X$, $X^*X \subseteq B$, $AX \subseteq X$. Then X is naturally a C^*-correspondence from A to B. Every C^*-correspondence can be represented in this form.

Example (C^*-correspondences vs. graphs). Let V, W be sets (spaces with discrete topology). Let $G = (E, s, r)$ be a graph from V to W, i.e., E is a set of edges and $s : E \to V$ and $r : E \to W$ are source and range maps. We define C^*-

correspondence X_G from $A = C_0(W)$ to $B := C_0(V)$ by putting $X_G := \{x \in C_0(E) : V \ni v \longmapsto \sum_{e\in s^{-1}(v)} |x(e)|^2 \in \mathbb{C}$ is in $C_0(V)\}$, and

$$\langle x, y\rangle_A(v) := \sum_{e\in s^{-1}(v)} \overline{x(e)}y(e),$$

$$(a\cdot x)(e) := a(r(e))x(e), \qquad (x\cdot b)(e) := x(e)b(s(e)).$$

Every C^*-correspondence from $C_0(W)$ to $C_0(V)$ is of this form.

If $A \xrightarrow{X} B$ and $B \xrightarrow{Y} C$ are C^*-correspondences then there is a C^*-correspondence $A \xrightarrow{X\otimes_B Y} C$ defined as follows. The space $X \otimes_B Y = \overline{\text{span}}\{x\otimes y : x \in X, y \in Y\}$ is the Hausdorff completion of the algebraic tensor product of X and Y with respect to the seminorm defined by the C-valued sesqui-linear form given by the formula

$$\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle_C := \langle y_1, \langle x_1, x_2\rangle_B \cdot y_2\rangle_C.$$

The left and right action on $X \otimes_B Y$ is defined in an obvious way: $a\cdot(x\otimes y)\cdot c := (a\cdot x)\otimes(y\cdot c)$ for $x \in X, y \in Y$, $a \in A$, $c \in C$. The C^*-correspondence $X \otimes_B Y$ is usually called the (*inner*) *tensor product* of X and Y. We encourage to think of it as a "composition" of C^*-correspondences X and Y.

Example (Induced representations). If $A \xrightarrow{X} B$ is a C^*-correspondence and $B \xrightarrow{H_\pi} \mathbb{C}$ is a representation of B, then $A \xrightarrow{X\otimes_B H_\pi} \mathbb{C}$ is a representation of A.

The latter representation is called the induced representation from π by X.

Example (Composition of homomorphisms). If $\alpha : A \to B$ and $\beta : B \to C$ are $*$-homomorphisms, then the C^*-correspondence $X_\alpha \otimes_B X_\beta$ is naturally isomorphic to the C^*-correspondence $X_{\beta\circ\alpha}$ associated to the $*$-homomorphism $\beta\circ\alpha : A \to C$.

Example (Concrete tensor products). Let $A, B, C \subseteq B(H)$ and $X, Y \subseteq B(H)$ be concrete C^*-correspondences $A \xrightarrow{X} B$ and $B \xrightarrow{Y} C$. Then $\overline{XY} = \overline{\text{span}}\{xy : x \in X, y \in Y\} \subseteq B(H)$ is a concrete C^*-correspondence $A \xrightarrow{\overline{XY}} C$ which is naturally isomorphic to the C^*-correspondence $X \otimes_B Y$.

Example (Composition of graphs). Let $G = (E, s, r)$ a graph from V to W and $H = (F, s, r)$ a graph from W to U. We define the composite graph $H \circ G := (F \circ E, s, r)$, where $F \circ E := \{(f, e) \in F \times E : s(f) = r(e)\}$, $s(f, e) := s(e)$ and $r(f, e) = r(f)$. Then we have a natural isomorphism of C^*-correspondences $X_H \otimes_B X_G \cong X_{H\circ G}$.

Let us consider a "category" whose objects are C^*-algebras and morphisms are non-degenerate C^*-correspondences. Strictly speaking such a structure is not a category, but a bicategory because the associativity holds only up to a natural isomorphism. More specifically, if $A \xrightarrow{X} B$, $B \xrightarrow{Y} C$ and $C \xrightarrow{Z} D$ are C^*-categories, we have a natural isomorphism:

$$X \otimes_B (Y \otimes_C Z) \cong (X \otimes_B Y) \otimes_C Z.$$

C^*-algebras treated as Hilbert modules act as "identity morphisms": we have $X \otimes_B B \cong X$ and $(A \otimes_A X) \cong X$ (here is where we use non-degeneracy of X). In particular, a C^*-correspondence $A \xrightarrow{X} B$ is "invertible" if there is a C^*-correspondence $B \xrightarrow{X^\star} A$ such that

$$X^\star \otimes_A X \cong B, \qquad X \otimes_B X^\star \cong A.$$

A C^*-correspondence is "invertible" in the above sense if and only if it is a Morita–Rieffel equivalence bimodule – an object that we describe below.

4. The Morita–Rieffel equivalence

Let A, B be C^*-algebras. A *Hilbert A-B-bimodule* is a space X which is both a right Hilbert B-module and a left Hilbert A-module such that the respective inner products satisfy

$${}_A\langle x, y\rangle z = x\langle y, z\rangle_B, \qquad x, y, z \in X.$$

Then $\langle X, X\rangle_B := \overline{\mathrm{span}}\{\langle x, y\rangle_B : x, y \in X\}$ is an ideal in B and ${}_A\langle X, X\rangle := \overline{\mathrm{span}}\{{}_A\langle x, y\rangle : x, y \in X\}$ is an ideal in A. We say that X is a (Morita–Rieffel) *equivalence bimodule* if in addition $\langle X, X\rangle_B = B$ and ${}_A\langle X, X\rangle = A$. If X is a Hilbert A-B-bimodule, and $X^\star$ is the adjoint Hilbert B-A-bimodule[2], then $X \otimes_B X^\star \cong \langle X, X\rangle_B$ and $(X^\star \otimes_A X) \cong {}_A\langle X, X\rangle$. Thus X is an equivalence bimodule if and only if it is "invertible". Two C^*-algebras A and B are *Morita equivalent* if there exists an equivalence Hilbert A-B-bimodule.

Remark. Every Hilbert A-B-bimodule X restricts to an equivalence ${}_A\langle X, X\rangle$-$\langle X, X\rangle_B$-bimodule. Every Hilbert A-B-bimodule X is a C^*-correspondence from A to B (the left action of A on X is necessarily given by adjointable operators). A C^*-correspondence $A \xrightarrow{X} B$ is a Hilbert A-B-bimodule if and only if the left action ϕ_X restricts to an isomorphism from an ideal J in A onto $\mathcal{K}(X)$ (then we necessarily have $\langle x, y\rangle_B = \phi_X|_J^{-1}(\Theta_{x,y})$).

Example (Compact operators). Every right Hilbert B-module is an equivalence $\mathcal{K}(X)$-$\langle X, X\rangle_B$-bimodule where ${}_{\mathcal{K}(X)}\langle x, y\rangle := \Theta_{x,y}$, $x, y \in X$. In particular, every Hilbert space H gives Morita equivalence between $\mathbb{C}$ and $K(H)$.

Example (Hereditary subalgebras and ideals). Let p be an element of a C^*-algebra C. The right ideal $X := pC$ is an equivalence bimodule establishing Morita equivalence between $A := pCp$ and $B := CpC$.

Example (Ternary rings of operators). A closed linear space $X \subseteq B(H)$ satisfying $XX^*X \subseteq X$ is called a (concrete) *ternary ring of operators.* Any such X is an equivalence bimodule from $A := \overline{XX^*}$ to $B := \overline{X^*X}$. Every equivalence A-B-bimodule can be represented in this form.

[2]It arises by exchanging the left and right structures.

Suppose that C^*-algebras A and B are embedded as corners into a C^*-algebra C, i.e., we have the decomposition $C = \left(\begin{smallmatrix} A & X \\ X^* & B \end{smallmatrix} \right)$. Then the space X with operations inherited from C is a Hilbert A-B-bimodule. It is an equivalence bimodule if and only if A and B are full C^*-subalgebras of C (i.e., we have $CAC = C$ and $CBC = C$). In fact, any two C^*-algebras A and B are Morita equivalent if and only if they can be embedded into a C^*-algebra C as full and complementary corners, see [1]. The celebrated theorem of Brown, Green and Rieffel [1] states the following:

Theorem 1. *If A and B have countable approximate units then A and B are Morita equivalent if and only if A and B are stably isomorphic, i.e., $A \otimes K(H) \cong B \otimes K(H)$ where H is a Hilbert space.*

Morita equivalent C^*-algebras A and B share a vast list of properties, cf. [5]. For instance, they have: isomorphic lattices of ideals $\mathrm{Ideal}(A) \cong \mathrm{Ideal}(B)$; homeomorphic spectra $\widehat{A} \cong \widehat{B}$ (equivalence classes of irreducible representations equipped with Jacobson topology); isomorphic K-groups $K_i(A) \cong K_i(B)$, $i = 0, 1$. Moreover, A is nuclear (resp., liminal or postliminal) if and only if B is nuclear (resp., liminal or postliminal).

Comments on actions of C^*-correspondences: Group actions of Hilbert bimodules on C^*-algebras correspond to Fell bundles over groups. They generalize group actions by automorphisms and the associated crossed products model all group graded C^*-algebras. Inverse semigroup actions by Hilbert bimodules can be viewed as noncommutative groupoids. They model all regular C^*-inclusions and in particular noncommutative Cartan pairs. Semigroup actions of C^*-correspondences correspond to product systems. They model various variants and generalizations of Cuntz-Pimsner algebras [4].

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